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# On Valuation Rings as Homomorphic Images of Valuation Domains.

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# ON VALUATION RINGS
# AS HOMOMORPHIC IMAGES OF
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## ABSTRACT

A number of years ago, I. Kaplansky raised informally the question of whether every valuation ring could be expressed as a homomorphic image of a valuation domain. By a valuation ring, we mean a commutative ring with identity whose ideals are linearly ordered by inclusion. The classical notion of a valuation ring included the assumption that the ring is a domain. For two cases an affirmative answer to Kaplansky's question is known: for 0-dimensional valuation rings; and, for valuation rings which are monoid rings.

In the early 40's, Kaplansky obtained structure theorems for a large class of (maximally complete) valuation domains. We approached his question with the idea of attempting to generalize his techniques in order to obtain anologous structure theorems for valuation rings. The hope was that, as a by-product, these theorems would yield the required valuation domain and homomorphism. In this paper are the results and questions which were discovered in the search for these structure theorems.

In Chapter 1, several definitions (e.g., immediate extension, maximal completion, pseudo-convergent sequence) and results are extended from valuation domains to valuation rings and, when appropriate, more generally to quasi-local rings. In particular, it is shown that a valuation ring which is maximal is also maximally

complete. In Chapter 2, after generalizing the concept of value group to that of value monoid, we define a long power series ring for such a monoid, with coefficients in a field, and prove such a ring is a homomorphic image of a valuation domain. In Chapter 3, we obtain a bound on the cardinality of a valuation ring which depends only on the cardinalities of the residue field and value monoid, then show every valuation ring has a maximal completion. Kaplansky's work on maximally complete valuation domains (valued fields) is outlined in Chapter 4. A discussion of what had been hoped for and remarks on several unanswered questions, along with miscellaneous results, conclude Chapter 4.

# INTRODUCTION

Several years ago, I. Kaplansky raised informally the question of whether every valuation ring could be expressed as a homomorphic image of a valuation domain. By a valuation ring, we mean a commutative ring with identity whose ideals are linearly ordered by inclusion. The classical notion of a valuation ring included the assumption that the ring be a domain. When we are concerned with a ring without 0-divisors, we shall use the term "domain".

For two cases an affirmative answer to Kaplansky's question is known: 1.) for 0-dimensional noetherian valuation rings [4, p. 545] (also [7, Thm. 3.3]); 2.) for valuation rings which are monoid rings [9]. It should be pointed out that a counterexample to Kaplansky's question claimed in [1, Ex. 10 (f), p. 453] directly contradicts Theorem 2.2 of [9].

In the early 40's, Kaplansky was able to obtain structure theorems for a large class of (maximally complete) valuation domains. We approached his question with the idea of trying to generalize his techniques in order to obtain analogous structure theorems for valuation rings. The hope was that these theorems would yield, as a by-product, the required valuation domain and homomorphism. What follows are the results and questions which were discovered in the search for these structure theorems.

In Chapter 1, several definitions (e.g., immediate extension, maximal completion, pseudo-convergent sequence) and results are extended from valuation domains to valuation rings and, when appropriate, more generally to quasi-local rings. In Chapter 2, generalizing the concept of value group to that of value monoid, we define a long power series ring for such a monoid, with coefficients in a field, and prove such a ring is a homomorphic image of a valuation domain. After establishing a bound on the cardinality of a valuation ring which depends only on the cardinalities of the residue field and value monoid, we show in Chapter 3 that every valuation ring has a maximal completion. An outline of Kaplansky's work on maximally complete valuation domains (valued fields) is presented in Chapter 4. A discussion of what had been hoped for and remarks on several unanswered questions, along with miscellaneous results, are also included in Chapter 4.

# CHAPTER I

## IMMEDIATE EXTENSIONS

By a monoid, we mean a commutative semigroup with identity. All groups are commutative; all rings are commutative with identity. If $R$ is a ring, then $\mathcal{U}(R)$ will denote the units of $R$ and $R^*$ the set of nonzero elements of $R$. For proper containment, we will use $\subset$.

The first step in trying to generalize Kaplansky's work is to decide what is meant by an immediate extension of a valuation ring. The replacement of the value group of a valuation domain by the value monoid of a valuation ring is needed.

If $R$ is a ring, then the principal ideals of $R$ form a monoid, denoted $(R)$, under ideal multiplication. We shall call $(R)$, written additively, the <u>semi-value monoid</u> of $R$. We can define a map $u\colon R \longrightarrow (R)$ by $u(r) = (r)$ for each element $r$ of $R$. The $(0)$ of $(R)$ is denoted by $\infty$. Partially order $(R)$ by reverse set inclusion, i.e., $(a) \leq (b)$ if and only if $(b) \subseteq (a)$. Then for any pair of elements $\alpha$ and $\beta$ of $(R)$, $\alpha \leq \beta$ if and only if there exists $\gamma$ from $(R)$ such that $\alpha + \gamma = \beta$. $R$ is a valuation ring if and only if this ordering is total, in which case $(R)$ is called the <u>value monoid</u> of $R$. Also, $R$ is a domain if and only if $(R)$ has the property that $(a) + (b) = \infty$, for $(a)$ and $(b)$ elements of $(R)$, implies at least one of $(a), (b)$ equals $\infty$.

Finally, $R$ is a quasi-local ring (meaning $R$ has a unique maximal ideal) if and only if $u$ is additive, i.e., $u(a) < u(b)$ implies $u(a + b) = u(a)$ where $a$ and $b$ are elements of $R$. Since the proof of the "if" direction of the last sentence is not immediate, we include it for the benefit of the reader. Assume $a$ and $b$ are nonunits of $R$. If $a + b \in \mathcal{U}(R)$, then $(-a)R \subset (a + b)R$, i.e., $u(a + b) < u(-a)$. By hypothesis, $u(a + b - a) = u(a + b)$. Thus, $b \in \mathcal{U}(R)$ which contradicts the choice of $b$. Therefore, $a + b$ is a nonunit, and hence, the set of nonunits of $R$ forms a maximal ideal.

Now let $D$ be a domain with quotient field $K$, and let $w\colon K^* \to K^*/\mathcal{U}(D)$ be the canonical map. The group $G = K^*/\mathcal{U}(D)$ may be ordered by defining the positive elements of $G$ to be $w(D^*)$, and then $w$ has the following properties: For all $a, b \in K^*$,

$$w(ab) = w(a) + w(b)$$

$$w(a + b) \geq \inf_G\{w(a), w(b)\} \quad \text{if} \quad a + b \in K^*$$

$$w(-1) = 0.$$

$D$ will be equal to $\{d \in D\colon w(d) \geq 0\} \cup \{0\}$; and $G$ is called the <u>semi-value group</u> of $D$. (See [8].) $D$ is a valuation domain if and only if $G$ is totally ordered. Moreover, if $G$ is totally ordered, then $G$ is called the <u>value group</u> of $D$. Again, $D$ is quasi-local if and only if $w$ is additive. [8, p. 579].

By making use of the semi-value monoid, it is now possible to extend the definition of immediate extensions of valuation domains to the case of quasi-local rings. Suppose $R$ and $R'$ are quasi-local rings with maximal ideals $\mathcal{M}$ and $\mathcal{M}'$, residue fields $k$ and $k'$, and semi-value monoids $M$ and $M'$. $R'$ is an extension of $R$ if

$R \subseteq R'$ and $M' \cap R = M$. If $R'$ is an extension of $R$, then we have the following diagrams:

$$\begin{tikzcd} R \arrow[r, dash] \arrow[d, dash] & R' \arrow[d, dash] \\ k \arrow[r, dash] & k' \end{tikzcd} \qquad\qquad \begin{tikzcd} R \arrow[r, dash] \arrow[d, dash] & R' \arrow[d, dash] \\ M \arrow[r, dash] & M' \end{tikzcd}$$

where these are canonical mappings of the residue fields and semi-value monoids. The canonical mapping $k \longrightarrow k'$ is an embedding, while the mapping $M \longrightarrow M'$ is an embedding if $R$ is a valuation ring, but not in general. If these mappings are also bijections, we shall write $k = k'$ and $M = M'$. We then say that $R'$ is an <u>immediate extension</u> of $R$. $R$ is called <u>maximally complete</u> if $R$ has no proper immediate extensions. $R'$ is called a <u>maximal completion</u> of $R$ if $R'$ is an immediate extension of $R$ and $R'$ is maximally complete.

Note that to say the canonical mapping $M \longrightarrow M'$ is an injection means that whenever $a$ and $b$ are elements of $R$ with $a = bs'$ for some $s' \in \mathcal{U}(R')$, then $a = bs$ for some $s \in \mathcal{U}(R)$. The mapping is a surjection if whenever $a'$ is an element of $R'$, then there exist $s' \in \mathcal{U}(R')$ and $a \in R$ such that $a' = as'$. To say that $k \longrightarrow k'$ is a surjection means if $s'$ is an element of $\mathcal{U}(R')$, then there exist $s \in \mathcal{U}(R)$ and $m' \in M'$ such that $s' = s + m'$.

The following lemma shows that the canonical map $M \longrightarrow M'$ preserves the order on $M$, i.e., $(b) <_M (a)$ implies $(b) <_{M'} (a)$.

<u>Lemma 1.1</u>. Assume $R'$ and $R$ are quasi-local rings with $R'$ an extension of $R$. If $a$ and $b$ are elements of $R$ with $aR \subset bR$, then $aR' \subset bR'$.

Proof: Clearly, $aR' \subseteq bR'$. If $aR' = bR'$, then there is an element $s'$ of $U(R')$ such that $b = as'$. Since $aR \subset bR$, $a = br$ for some $r \in R \setminus U(R)$. Also, $r \notin U(R')$ because $M' \cap R = M$. Thus, $b = brs'$, which implies $b(1 - rs') = 0$. Since $1 - rs'$ is a unit of $R'$, $b = 0$. But this contradicts the choice of $b$, so $aR' \neq bR'$. ##

Remark 1.2. If the canonical map $M \longrightarrow M'$ is a bijection, then by Lemma 1.1 and the following Lemma 1.3, order is preserved in both directions under $M \longrightarrow M'$.

Lemma 1.3. Assume $R'$ and $R$ are quasi-local rings with $R'$ an extension of $R$ and the canonical map $M \longrightarrow M'$ a bijection. If for elements $a$ and $b$ of $R$, $aR' \subset bR'$, then $aR \subset bR$.

Proof: $aR' \subset bR'$ implies $a = br'$ for some $r' \in R' \setminus U(R')$. Since $M \longrightarrow M'$ is onto, $r' = rs'$ for some $s' \in U(R')$ and $r \in R$. Note that $r \notin U(R)$. Since $aR' = brR'$ and $M \longrightarrow M'$ is an injection, $aR = brR$. Therefore, $aR \subset bR$. ##

In order to refer to Kaplansky's work, we require the following lemma:

Lemma 1.4. If $R$ is a valuation ring, then any immediate extension $R'$ of $R$ is a valuation ring. If $D$ is a valuation domain, then any immediate extension $D'$ of $D$ is a valuation domain. If $R$ is a quasi-local ring (domain) which is not a valuation ring (domain), then any immediate extension $R'$ of $R$ is not a valuation ring.

Proof: If $R'$ is an immediate extension of $R$, then by Remark 1.2, the canonical bijection $(R) \longrightarrow (R')$ preserves order in both directions. The (order) structure of $(R')$ yields the desired properties of $R'$. Likewise for $D'$. ##

In Section 3 of [5], it is shown that for certain valuation domains $D$, a maximal completion of $D$ is, in a certain sense, uniquely determined. With more restrictive conditions on $D$, any maximal completion of $D$ is (equivalent to) a long power series domain [5, Section 4]. See Chapter 2 of this paper for a definition of a long power series domain and Chapter 4 for a discussion of Kaplansky's work.

A set of elements $\{r_\rho\}_{\rho\varepsilon A}$, $A$ a well-ordered set, of a quasi-local ring $R$ is called a pseudo-convergent sequence if $A$ does not have a last element and $u(r_\sigma - r_\rho) < u(r_\tau - r_\sigma)$ for $\rho < \sigma < \tau$ where $\rho, \sigma, \tau \varepsilon A$. The next three lemmas are helpful in understanding the behavior of a pseudo-convergent sequence.

Lemma 1.5. Assume $\{r_\rho\}_{\rho\varepsilon A}$ is a pseudo-convergent sequence in a quasi-local ring $R$. If $u(r_\rho) = u(r_\sigma)$ for some $\rho < \sigma$, then $u(r_\tau) = u(r_\rho)$ for all $\rho < \tau$.

Proof: Since $u(r_\rho) = u(r_\sigma)$, $r_\rho = br_\sigma$ where $b \varepsilon \mathcal{U}(R)$. Consider an arbitrary $\tau \varepsilon A$ with $\rho < \tau$.

Case 1. $\rho < \sigma < \tau$.

Since $u(r_\sigma - r_\rho) < u(r_\tau - r_\sigma)$ means $(r_\tau - r_\sigma)R \subset (r_\sigma - r_\rho)R$, $r_\tau - r_\sigma = (r_\sigma - r_\rho)a$ for some $a \notin \mathcal{U}(R)$. So $r_\tau = (1 + a - ba)r_\sigma$. Since $R$ is quasi-local, $1 + a - ba \varepsilon \mathcal{U}(R)$. Thus, $u(r_\tau) = u(r_\sigma)$.

<u>Case 2</u>. $\rho < \tau < \sigma$ .

Since $u(r_\tau - r_\rho) < u(r_\sigma - r_\tau)$, we have $r_\sigma - r_\tau = a(r_\tau - r_\rho)$ for some $a \notin U(R)$. Substituting $br_\sigma$ for $r_\rho$, the equality can be transformed to $(1 + ab)r_\sigma = (1 + a)r_\tau$ . Since $R$ is quasi-local, both $1 + ab$, $1 + a \in U(R)$. Therefore, $u(r_\tau) = u(r_\sigma) = u(r_\rho)$. ##

<u>Lemma 1.6</u>. If $\{r_\rho\}_{\rho\varepsilon A}$ is a pseudo-convergent sequence in a quasi-local ring $R$, then $u(r_\sigma - r_\rho) = u(r_{\rho+1} - r_\rho)$ for all $\rho < \sigma$.

<u>Proof</u>: The proof is identical to the one in the valuation domain case [11, p. 39]. ##

In the case of valuation rings, Lemma 1.5 can be rephrased as:

<u>Lemma 1.7</u>. If $\{r_\rho\}_{\rho\varepsilon A}$ is a pseudo-convergent sequence in a valuation ring $R$, then either

i.) $u(r_\rho) < u(r_\sigma)$ for all $\rho < \sigma$ or

ii.) $u(r_\rho) = u(r_\sigma)$ for all $\rho, \sigma > \lambda$ for some $\lambda$.

<u>Proof</u>: See [11, p. 39]. ##

For each $\rho \varepsilon A$, because of Lemma 1.6, set $\gamma_\rho = u(r_{\rho+1} - r_\rho) = u(r_\sigma - r_\rho)$ for $\rho < \sigma$. An element $r$ of $R$ is called a <u>pseudo-limit</u> of $\{r_\rho\}_{\rho\varepsilon A}$ if $u(r - r_\rho) = \gamma_\rho$ for all $\rho$. $R$ is called <u>maximal (1)</u> if every pseudo-convergent sequence in $R$ has a pseudo-limit in $R$.

The above definition of pseudo-convergence generalizes the one used by Kaplansky [5, p. 303] and Schilling [11, p. 39]. However, it should be noted that Ostrowski's treatment of pseudo-convergence differs slightly in that $u(r_\sigma - r_\rho) < u(r_\tau - r_\sigma)$ is required from some point on. [10, p. 368].

A ring $R$ is called maximal (2) if every system of congruences $\{X \equiv r_\rho \bmod I_\rho\}_{\rho \in A}$ which is finitely solvable in $R$ has a solution in $R$ (where the $r_\rho$ are elements of $R$, the $I_\rho$ are ideals of $R$, and $A$ is any index set).

The equivalence of maximal (1) and maximal (2) for valuation rings is demonstrated below; see Corollary 1.13. However, by making use of two difficult theorems, 1.8 and 1.9 below, the equivalence in the case of valuation domains is a triviality.

Theorem 1.8. Assume $D$ is a valuation domain. $D$ is maximal (1) if and only if $D$ is maximally complete.

Proof: [5, Thm. 4]. ##

Theorem 1.9. Assume $D$ is a valuation domain. $D$ is maximal (2) if and only if $D$ is maximally complete.

Proof: [2, Thm. 12.6]. ##

We need several lemmas before we are able to show the equivalence of maximal (1) and maximal (2) for valuation rings.

Lemma 1.10. Assume $\{X \equiv r_\rho \bmod I_\rho\}_{\rho \in A}$ is a pairwise solvable system of congruences in a ring $R$. If $I_\tau \subset I_\sigma$, then $r_\tau \equiv r_\sigma \bmod I_\sigma$. In particular, $r_\tau + I_\tau \subset r_\sigma + I_\sigma$.

Proof: There exists, by hypothesis, an element $r$ of $R$ such that $r \equiv r_\sigma \bmod I_\sigma$ and $r \equiv r_\tau \bmod I_\tau$. But $I_\tau \subset I_\sigma$ implies

$r \equiv r_\tau \bmod I_\sigma$. Therefore, $r_\tau \equiv r_\sigma \bmod I_\sigma$. ##

Lemma 1.11. In an arbitrary ring $R$, if the system of congruences $\{X \equiv r_\rho \bmod I_\rho\}_{\rho\varepsilon A}$ is pairwise solvable and $\{I_\rho\}_{\rho\varepsilon A}$ forms a chain, then the system is finitely solvable.

Proof: Consider $X \equiv r_i \bmod I_i$ , $i = 1, 2, \ldots , n$ where, by relabeling if necessary, $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n$. Now $r_1$ is a solution to this finite system since, by Lemma 1.10, $r_1 \equiv r_i \bmod I_i$ , $i = 2, \ldots , n$. ##

Proposition 1.12.

i.) Suppose $R$ is a quasi-local ring. If $R$ is maximal (2), then $R$ is maximal (1).

ii.) Assume $R$ is a valuation ring. If $R$ is maximal (1), then $R$ is maximal (2).

Proof of i.): To show $R$ is maximal (1), we must find a pseudo-limit in $R$ for any pseudo-convergent sequence $\{r_\rho\}_{\rho\varepsilon A}$ in $R$. First, we construct a finitely solvable (and hence solvable, since $R$ is maximal (2)) system of congruences.

For each $\rho \varepsilon A$, let $I_\rho = \{a \varepsilon R\colon\ u(a) \geq \gamma_\rho\}$. Recall that $\gamma_\rho = u(r_{\rho+1} - r_\rho)$. If $a$ and $b$ are elements of $I_\rho$, then $u(a), u(b) \geq \gamma_\rho$ implies $u(a + b) \geq \gamma_\rho$; if $a \varepsilon I_\rho$, then $u(ra) \geq u(a) \geq \gamma_\rho$ for all elements $r$ of $R$. Thus, $I_\rho$ is an ideal of $R$. Also, $\{I_\rho\}_{\rho\varepsilon A}$ forms a chain. Indeed, if $\sigma$ and $\tau$ are elements of $A$ with $\sigma < \tau$, then $u(a) \geq \gamma_\tau > \gamma_\sigma$ for all elements $a$ of $I_\tau$, i.e., $I_\tau \subseteq I_\sigma$ for $\sigma < \tau$.

Consider $X \equiv r_\tau \bmod I_\tau$ and $X \equiv r_\sigma \bmod I_\sigma$ where $\sigma < \tau$. Note that $u(r_\tau - r_\sigma) = u(r_{\sigma+1} - r_\sigma)$ by Lemma 1.6. Thus, $r_\tau \equiv r_\sigma \bmod I_\sigma$. Therefore, $\{X \equiv r_\rho \bmod I_\rho\}_{\rho\varepsilon A}$ is pairwise solvable and hence, by Lemma 1.11, finitely solvable. Since $R$ is maximal (2), there is a solution $r$ of the entire system. Then $u(r - r_\rho) \geq \gamma_\rho$ for all $\rho$. But $u(r - r_\rho) = u(r - r_{\rho+1} + r_{\rho+1} - r_\rho) = \gamma_\rho$ since $u(r - r_{\rho+1}) \geq \gamma_{\rho+1} > \gamma_\rho = u(r_{\rho+1} - r_\rho)$. Thus, $r$ is a pseudo-limit of $\{r_\rho\}_{\rho\varepsilon A}$ . ##

Proof of ii.): To show a valuation ring $R$ is maximal (2), we must find a solution to any finitely solvable system of congruences:

$$(*) \quad \{X \equiv r_\lambda \bmod I_\lambda\}_{\lambda\varepsilon A} .$$

Fix an element $\rho_0 \varepsilon A$ and let

$\mathcal{S} = \{\{r_\rho\}_{\rho\varepsilon S}$: $S \subseteq A$; $S$ is well-ordered with respect to an ordering where $(**)$ the first element of $S$ is $\rho_0$, and $(***)$ $\rho, \rho' \varepsilon S$ with $\rho < \rho'$ implies $I_{\rho'} \subset I_\rho$ and $r_\rho \not\equiv r_{\rho'} \bmod I_{\rho'}\}$. Note that $\{r_{\rho_0}\} \varepsilon \mathcal{S}$. Define a partial ordering on $\mathcal{S}$ by $\{r_\rho\}_{\rho\varepsilon S} \leq \{r'_{\rho'}\}_{\rho'\varepsilon S'}$ if and only if $S$ is an initial segment of $S'$ and $r_\rho = r_{\rho'}$ for all $\rho \varepsilon S$. Any chain $\mathcal{S}_0 = \{\{r_\rho\}_{\rho\varepsilon S_\delta}\}_{\delta\varepsilon B}$ of $\mathcal{S}$ has an upper bound in $\mathcal{S}$ by letting $S' = \bigcup_\delta S_\delta$, then forming $\{r_\rho\}_{\rho\varepsilon S'}$ . By Zorn's Lemma, $\mathcal{S}$ has a maximal element, $\mathcal{S}_1 = \{r_\rho\}_{\rho\varepsilon S_1}$.

If $\rho < \sigma < \tau$ are elements of $S_1$, then $I_\tau \subset I_\sigma \subset I_\rho$. Since the system $(*)$ is finitely solvable, by Lemma 1.10 $r_\tau \equiv r_\sigma \bmod I_\sigma$. Since $S_1$ satisfies $(**)$, $r_\rho \not\equiv r_\sigma \bmod I_\sigma$. Thus, $r_\sigma - r_\rho \notin I_\sigma$ and $r_\tau - r_\sigma \varepsilon I_\sigma$. Since $R$ is a valuation ring, it follows that $u(r_\tau - r_\sigma) > u(r_\sigma - r_\rho)$.

Case 1. $S_1$ does not have a last element. Then the preceding paragraph shows that $\{r_\rho\}_{\rho \varepsilon S_1}$ is a pseudo-convergent sequence. Since $R$ is maximal (1), this sequence has a pseudo-limit $r$.

Suppose $\sigma \varepsilon A$ such that $I_\sigma \subset \cap\{I_\tau : \tau \varepsilon S_1\}$. Then we have:

Claim: $r_\tau \not\equiv r_\sigma \bmod I_\sigma$ for all $\tau \varepsilon S_1$.

Proof of Claim: Assume there exists $\tau \varepsilon S_1$ such that $r_\tau \equiv r_\sigma \bmod I_\sigma$; we shall show this assumption leads to a contradiction. Since $(*)$ is finitely solvable, $r_\sigma \equiv r_\rho \bmod I_\rho$ for all $\rho \varepsilon S_1$ by Lemma 1.10. We may choose $j \varepsilon S_1$ with $j > \tau$ since $S_1$ does not have a last element. Then $u(r_\sigma - r_j) = u(r_\sigma - r_\tau + r_\tau - r_j) = u(r_\tau - r_j)$ since $r_\sigma - r_\tau \varepsilon I_\sigma \subseteq I_j$ and $r_\tau \not\equiv r_j \bmod I_j$ implies $u(r_\sigma - r_\tau) > u(r_\tau - r_j)$. The equality $u(r_\sigma - r_j) = u(r_\tau - r_j)$ means $(r_\sigma - r_j)R = (r_\tau - r_j)R$, which contradicts $r_\sigma - r_j \varepsilon I_j$ and $r_\tau - r_j \notin I_j$. Hence, the claim is proved.

We are now able to show $r$ (a pseudo-limit of $\{r_\rho\}_{\rho \varepsilon S_1}$) is a solution of $(*)$.

First of all, we observe that $r \equiv r_\lambda \bmod I_\lambda$ for any $\lambda \varepsilon S_1$. Because if $\lambda \varepsilon S_1$, since $r$ is, by definition, a pseudo-limit of $\{r_\rho\}_{\rho \varepsilon S_1}$, $u(r - r_\lambda) = u(r_{\lambda+1} - r_\lambda)$. By Lemma 1.10, $r_{\lambda+1} \equiv r_\lambda \bmod I_\lambda$. Thus, $(r - r_\lambda)R = (r_{\lambda+1} - r_\lambda)R \subseteq I_\lambda$ implies $r \equiv r_\lambda \bmod I_\lambda$.

Second, we observe that if $\sigma \varepsilon A \smallsetminus S_1$ and $I_\rho \subseteq I_\sigma$ for some $\rho \varepsilon S_1$, then $r \equiv r_\sigma \bmod I_\sigma$ (because $r \equiv r_\rho \bmod I_\rho$ and $r_\rho \equiv r_\sigma \bmod I_\sigma$).

Third, we observe that if $\sigma \in A \smallsetminus S_1$, then we are in the case of the preceding paragraph, i.e., $I_\sigma \subseteq I_\rho$ for some $\rho \in S_1$. For suppose not, then $I_\sigma \subset \bigcap\{I_\tau: \tau \in S_1\}$. Let $S_2 = S_1 \cup \{\sigma\}$, ordered so that $S_1$ is an initial segment of $S_2$ and $\sigma$ is greater than every element of $S_1$. We now show $\mathcal{S}_2 = \{r_\rho\}_{\rho \in S_2}$ is an element of $\mathcal{S}$. Certainly, $S_2$ is well-ordered with first element $\rho_0$, so $(**)$ is satisfied. $S_1$ already satisfies $(***)$ while, by assumption, $I_\sigma \subset I_\tau$ for all $\tau \in S_1$, and, by the above claim, $r_\tau \not\equiv r_\sigma \bmod I_\sigma$ for all $\tau \in S_1$. Thus, $S_2$ satisfies $(***)$, and $\mathcal{S}_2 \in \mathcal{S}$. But $\mathcal{S}_1$ is a maximal element of $\mathcal{S}$ and $\mathcal{S}_1 < \mathcal{S}_2$, which is a contradiction. Hence, if $\sigma \in A \smallsetminus S_1$, then there is some $\rho \in S_1$ such that $I_\rho \subseteq I_\sigma$.

<u>Case 2</u>. $S_1$ has a last element, say $\tau$. We shall show $r_\tau$ is a solution of $(*)$. Consider an arbitrary congruence of $(*)$, $X \equiv r_\sigma \bmod I_\sigma$.

If $I_\tau \subseteq I_\sigma$, then by Lemma 1.10, $r_\tau \equiv r_\sigma \bmod I_\sigma$, and we are done.

For the other possbility, assume $I_\sigma \subset I_\tau$ and $r_\tau \not\equiv r_\sigma \bmod I_\sigma$. We shall reach a contradiction by constructing an element of $\mathcal{S}$ which is strictly greater than $\mathcal{S}_1$; namely, $\mathcal{S}_2 = \{r_\rho\}_{\rho \in S_2}$, where $S_2 = S_1 \cup \{\sigma\}$ is ordered so that $S_1$ is an initial segment of $S_2$, and $\sigma$ is greater than every element of $S_1$. $S_2$ satisfies $(**)$ since $S_2$ is well-ordered with first element $\rho_0$. We now show that $S_2$ satisfies $(***)$. Since $\tau$ is the last element of $S_1$, $I_\tau \subset I_\rho$ for all $\rho \in S_1 \smallsetminus \{\tau\}$. Thus, $I_\sigma \subset I_\rho$ for all $\rho \in S_1$. Observe that $r_\tau \not\equiv r_\sigma \bmod I_\sigma$, $r_\sigma \equiv r_\tau \bmod I_\tau$, and $r_\rho \not\equiv r_\tau \bmod I_\tau$ for all $\rho \in S_1 \smallsetminus \{\tau\}$ which implies $r_\rho \not\equiv r_\sigma \bmod I_\sigma$ for all

$\rho \in S_1$. $S_1$ satisfying $(***)$ along with $I_\sigma \subset I_\rho$ and $r_\rho \not\equiv r_\sigma \bmod I_\sigma$ for all $\rho \in S_1$ implies $S_2$ satisfies $(***)$. Thus, $S_2 \in \mathcal{S}$ and $S_2 > S_1$, which is a contradiction to the choice of $S_1$ as a maximal element of $\mathcal{S}$. Therefore, if $I_\sigma \subset I_\tau$, then $r_\tau \equiv r_\sigma \bmod I_\sigma$, and we are done. ##

Corollary 1.13. If R is a valuation ring, then R is maximal (1) if and only if R is maximal (2).

If R is a valuation ring, we will use the term "maximal" instead of maximal (1) or maximal (2).

We now present a partial generalization of Theorems 1.8 and 1.9 for valuation rings.

Lemma 1.14. If R' is a proper immediate extension of a valuation ring R, then there exists a system of congruences in R which is pairwise solvable in R (hence, finitely solvable in R), but not solvable in R.

Proof: By Lemma 1.4, R' is a valuation ring. Fix an element s' of $R' \setminus R$. Since $(R) = (R')$, $s'R' = sR$ for some $s \in R$. Then $s' = sb'$ for some $b' \in \mathcal{U}(R')$, and $s' \notin R$ implies $b' \notin R$. Thus, by replacing $s'$ by $b'$, we may further assume $s' \in \mathcal{U}(R')$. Let $T = \{u'(s' - r)\colon\ r \in R^*\}$; note that $\infty \notin T$. We first show that T does not have a greatest element. Let $r \in R^*$; we will show there is an element s of $R^*$ such that $u'(s' - s) > u'(s' - r)$. Since $(R) = (R')$, there is an element t of $R^*$ such that $u'(s' - r) = u'(t)$. There exists $a'$, an element of $\mathcal{U}(R')$, such that $s' - r = ta'$. Since $k = k'$,

$a' = a + m'$ where $a \in U(R)$ and $m' \in M'$. Thus, $s' - r - ta = tm'$ and $u'(s' - r) = u'(t) < u'(tm') = u'(s' - r - ta)$. Set $s = r + ta$. Note that $s \in R^*$, for if $s = 0$, then $s' = tm'$, i.e., $s'$ would be a nonunit of $R'$, which contradicts the choice of $s'$.

For each element $r$ of $R^*$, define $I_r = \{t \in R\colon\ u'(t) \geq u'(s' - r)\}$. We now show the system of congruences $\{X \equiv r \bmod I_r\}_{r\in R^*}$ is pairwise solvable in $R$, but not solvable in $R$. Consider $X \equiv r_1 \bmod I_{r_1}$ and $X \equiv r_2 \bmod I_{r_2}$. We may assume $u'(r_1 - s') \leq u'(r_2 - s')$. Since $u'(r_2 - r_1) = u'(r_2 - s' + s' - r_1) \geq u'(r_1 - s')$, we have $r_2 \equiv r_1 \bmod I_{r_1}$. Thus, $r_2$ is a solution to the given pair of congruences.

Now assume $r_0$ (an element of $R$) is a solution to the entire system of congruences, i.e., $u'(r_0 - r) \geq u'(s' - r)$ for all elements $r$ of $R^*$. This implies $u'(s' - r_0) \geq u'(s' - r)$ for all $r \in R^*$ (since $s' - r_0 = s' - r + r - r_0$). Hence, $T$ has a largest element (namely, $0$ if $r_0 = 0$, or $u'(s' - r_0)$ if $r_0 \in R^*$), which contradicts the opening paragraph of this proof. Therefore $\{X \equiv r \bmod I_r\}_{r\in R^*}$ is finitely solvable in $R$, but not solvable in $R$. ##

If $R'$ is a proper immediate extension of a valuation ring $R$, then the above proof shows that any element of $U(R')$, which is not an element of $R$, is a solution to an extended system of congruences from $R$ where the original system is pairwise solvable in $R$, but not solvable in $R$. (If $\{X \equiv r_\rho \bmod I_\rho\}_{\rho\in A}$ is a system of congruences in $R$, then the extended system in $R'$ is $\{X \equiv r_\rho \bmod I_\rho R'\}_{\rho \varepsilon A}$ .)

Theorem 1.15. A maximal valuation ring is maximally complete.

Proof: If $R$ is a valuation ring which is not maximally complete, then there exists a proper immediate extension $R'$ of $R$. By Lemma 1.14, there exists a system of congruences in $R$ which is finitely solvable in $R$, but not solvable in $R$, i.e., $R$ is not maximal. ##

# CHAPTER II

## LONG POWER SERIES RINGS

In this chapter, we define a long power series ring which is a generalization of the long power series domain found in [2, p. 82] and [11, p. 23]. First, we need more information concerning value monoids.

Remark 2.1. A monoid $M$ is naturally ordered if $M$ is partially ordered by: for elements $\alpha$ and $\beta$ of $M$, $\alpha \leq \beta$ if and only if there is an element $\gamma$ of $M$ such that $\alpha + \gamma = \beta$ [3, p. 631]. $M$ is called 0-segmental if i.) $M$ is naturally totally ordered, ii.) there is an element $\infty$ such that $\infty \geq \alpha$ for all $\alpha \in M$, and iii.) for elements $\alpha$ and $\beta$ of $M$, $\alpha + \beta = \beta$ implies $\alpha = 0$ or $\beta = \infty$ [12, p. 406]. Shores [12], using techniques of Clifford [3], proved that $M$ is a 0-segmental monoid (if and) only if there exists a totally ordered group $G$ and an ideal $I_\infty$ of $G^+ \cup \{\infty\}$ such that the Rees factor monoid $(G^+ \cup \{\infty\})/I_\infty$ is isomorphic to $M$. "Ideal" means $I_\infty \subseteq M$ and $I_\infty + M \subseteq I_\infty$; and "Rees factor monoid" means two elements $a$ amd $b$ of $G^+ \cup \{\infty\}$ are equivalent if and only if $a = b$ or both $a$ and $b$ are elements of $I_\infty$. The Rees factor monoid amounts to sending each element of $I_\infty$ to $\infty$ and every other element to itself. It is known classically that for any totally ordered group $G$, there exists a valuation domain D having $G^+ \cup \{\infty\}$ as its value monoid. Since the inverse image of $I_\infty$ under $u: D \longrightarrow G^+ \cup \{\infty\}$ is an ideal I of D, it follows that whenever a 0-segmental monoid M is given, there exists a valuation domain D which satisfies the diagram:

$$\begin{array}{ccc} D & \longrightarrow & (D) \\ \downarrow & & \downarrow \\ D/I = R & \longrightarrow & (R) \cong M. \end{array}$$

In particular, the 0-segmental monoids are precisely the value monoids of valuation rings. We will use the term "value monoid" instead of 0-segmental monoid.

2.2 Definition of K[[M]]: Let K be a field and M a value monoid. Denote by K[[M]] the set of all series of the form $\sum_{\alpha\in S} a_\alpha X^\alpha$ where $S \subseteq M \setminus \{\infty\}$ is well-ordered by the induced ordering of M and the $a_\alpha$ are elements of K. If $\ell = \sum_{\alpha\in S} a_\alpha X^\alpha$ is an element of K[[M]], then the support of $\ell$, denoted supp($\ell$), is defined by $\mathrm{supp}(\ell) = \{\alpha \in S: a_\alpha \neq 0\}$. Two series $\ell_1 = \sum_{\alpha\in S} a_\alpha X^\alpha$ and $\ell_2 = \sum_{\beta\in T} b_\beta X^\beta$ are said to be equal if $\mathrm{supp}(\ell_1) = \mathrm{supp}(\ell_2)$ and $\alpha \in \mathrm{supp}(\ell_1)$ implies $a_\alpha = b_\alpha$.

The sum of two series of K[[M]] is defined by $\sum_{\alpha\in S} a_\alpha X^\alpha + \sum_{\beta\in T} b_\beta X^\beta = \sum_{\rho\in S'} c_\rho X^\rho$ where $S' = S \cup T$ and

$$c_\rho = \begin{cases} a_\rho + b_\rho & \text{if } \rho \in S \cap T \\ a_\rho & \text{if } \rho \in S \setminus T \\ b_\rho & \text{if } \rho \in T \setminus S \end{cases}$$

The product is defined by

$$\left(\sum_{\alpha\in S} a_\alpha X^\alpha\right)\left(\sum_{\beta\in T} b_\beta X^\beta\right) = \sum_{\rho\in W}\left(\sum_{\alpha+\beta=\rho} a_\alpha b_\beta\right)X^\rho \quad \text{where}$$

$W = \{\rho\colon \rho = \alpha + \beta < \infty,\ \alpha \in S,\ \beta \in T\}$, if this set $W$ is non-empty. If, on the other hand, $W = \phi$, i.e., $\alpha + \beta = \infty$ for all $\alpha \in S$, $\beta \in T$, then the product is defined to be $0$.

In order to verify that the product of two long power series is an element of $K[[M]]$, it suffices to show: i.) $\sum_{\alpha+\beta=\rho} a_\alpha b_\beta$ is a finite sum; and ii.) $W$ is well-ordered.

Proof of i.): Fix an element $\rho$ of $W$ and suppose $\{(\alpha,\beta) \in S\text{x}T\colon\ \alpha + \beta = \rho\}$ is an infinite set. Let $A = \{\alpha \in S;$ there exists $\beta \in T$ such that $\alpha + \beta = \rho\}$. Let $\alpha_1$ be the least element of $A$. Let $\alpha_2$ be the least element of $A \smallsetminus \{\alpha_1\}$, etc.; thus, a sequence $\{\alpha_i\}_{i\in N} \subseteq A$ is obtained with $\alpha_1 < \alpha_2 < \ldots$ . This yields $\{\beta_i\}_{i\in N} \subseteq T$ with $\beta_1 > \beta_2 > \ldots$ , by letting $\beta_i$ denote the element of $T$ such that $\alpha_i + \beta_i = \rho$. But $\{\beta_i\}$ is a subset of $T$ without a least element, a contradiction to $T$ being well-ordered. Thus, $\{(\alpha,\beta) \in S\text{x}T\colon\ \alpha + \beta = \rho\}$ must be finite.

Proof of ii): If a nonempty subset $C$ of $W$ does not have a least element, then there exists a sequence $\{\rho_i\}_{i\in N} \subseteq C$ with $\rho_1 > \rho_2 > \ldots$ . For each $i$, pick an element $\alpha_i$ of $S$ and an element $\beta_i$ of $T$ so that $\alpha_i + \beta_i = \rho_i$. Let $\alpha_{i_1}$ be the least element of $\{\alpha_i\}$. Let $\alpha_{i_2}$ be the least element of $\{\alpha_{i_1+j}\}_{j\in N}$ , etc., to define $\alpha_{i_1} \leq \alpha_{i_2} \leq \cdots$ , with $i_1 < i_2 < \cdots$ . We have defined two subsequences $\{\alpha_{i_j}\}_{j\in N} \subseteq S$ and $\{\beta_{i_j}\}_{j\in N} \subseteq T$. Since $\alpha_{i_j} \leq \alpha_{i_{j+1}}$, $\rho_{i_{j+1}} < \rho_{i_j}$ and $\alpha_{i_j} + \beta_{i_j} = \rho_{i_j}$, we have $\beta_{i_j} > \beta_{i_{j+1}}$ for all $j$. Thus, $\{\beta_{i_j}\}$ does not have a least element, which is a contradiction to $T$ being well-ordered.

Note that $K$ is contained in $K[[M]]$ by means of the identification $a \longmapsto aX^0$.

<u>Proposition 2.3</u>. If $M$ is a value monoid and $K$ a field, then $K[[M]]$ (with the above defined operations of sum and product) forms a ring.

<u>Proof</u>: All is easy except for the associative law for multiplication and showing multiplication distributes over addition. Assume that $\ell_1 = \sum_{\alpha\varepsilon S} a_\alpha X^\alpha$, $\ell_2 = \sum_{\beta\varepsilon T} b_\beta X^\beta$ and $\ell_3 = \sum_{\gamma\varepsilon W} c_\gamma X^\gamma$ are elements of $K[[M]]$. Now $(\ell_1\ell_2)\ell_3 = \left(\sum_{\rho\varepsilon S+T} \left(\sum_{\alpha+\beta=\rho} a_\alpha b_\beta\right)X^\rho\right)\ell_3$

$$= \sum_{\sigma\varepsilon(S+T)+W} \left(\sum_{\rho+\gamma=\sigma} \left(\sum_{\alpha+\beta=\rho} a_\alpha b_\beta\right) c_\gamma\right)X^\sigma$$

where $S + T = \{\rho\colon\ \rho = \alpha + \beta < \infty,\ \alpha \varepsilon S,\ \beta \varepsilon T\}$ and $(S + T) + W = \{\sigma\colon\ \sigma = \rho + \gamma < \infty,\ \rho \varepsilon S + T,\ \gamma \varepsilon W\}$. Note that $(S + T) + W = \phi$ if and only if $S + (T + W) = \phi$, in which case $(\ell_1\ell_2)\ell_3 = 0 = \ell_1(\ell_2\ell_3)$. So assume both sets are nonempty, but then they are equal, for:

$\sigma \varepsilon (S + T) + W \iff \sigma = \rho + \gamma < \infty$ where $\rho \varepsilon S + T$ and $\gamma \varepsilon W$

$\iff \sigma = \alpha + \beta + \gamma < \infty$ where $\gamma \varepsilon W$ and

$$\rho = \alpha + \beta < \infty \text{ where } \alpha \in S \text{ and } \beta \in T$$
$$\Leftrightarrow \sigma = \alpha + \rho' < \infty \text{ where } \rho' \in T + W \text{ and } \alpha \in S$$
$$\Leftrightarrow \sigma \in S + (T + W).$$

If $\sigma$ is a fixed element of $(S + T) + W$, then for $(\ell_1\ell_2)\ell_3$ the coefficient of the $X^\sigma$ term is

$$\sum_{\rho+\gamma=\sigma} \Big( \sum_{\alpha+\beta=\rho} a_\alpha b_\beta \Big) c_\gamma = \sum_{\rho+\gamma=\sigma} \Big( \sum_{\alpha+\beta=\rho} a_\alpha b_\beta c_\gamma \Big) = \sum_{\alpha+\beta+\gamma=\sigma} a_\alpha b_\beta c_\gamma$$
$$= \sum_{\sigma=\gamma+\rho'} \Big( \sum_{\rho'=\beta+\alpha} a_\alpha b_\beta c_\gamma \Big) = \sum_{\sigma=\alpha+\rho'} a_\alpha \Big( \sum_{\rho'=\beta+\gamma} b_\beta c_\gamma \Big)$$

which is the coefficient of the $X^\sigma$ term for $\ell_1(\ell_2\ell_3)$. Thus, $(\ell_1\ell_2)\ell_3 = \ell_1(\ell_2\ell_3)$.

Now consider $\ell_1(\ell_2 + \ell_3) = \ell_1\Big( \sum_{\rho\in T\cup W} d_\rho X^\rho \Big)$

$$= \sum_{\sigma\in S+(T\cup W)} \Big( \sum_{\sigma=\alpha+\rho} a_\alpha d_\rho \Big) X^\sigma$$

(where $S + (T \cup W) = \{\sigma\colon \ \sigma = \alpha + \rho < \infty,\ \alpha \in S,\ \rho \in T \cup W\}$)

$$= \sum_{\alpha\in S+(T\cup W)} \Big( \sum_{\rho\in T\cap W} a_\alpha(b_\rho + c_\rho) + \sum_{\rho\in T\smallsetminus W} a_\alpha b_\rho + \sum_{\rho\in W\smallsetminus T} a_\alpha c_\rho \Big) X^\sigma$$
$$= \sum_{\sigma\in S+(T\cup W)} \Big( \sum_{\rho\in T} a_\alpha b_\rho + \sum_{\rho\in W} a_\alpha c_\rho \Big) X^\sigma$$
$$= \sum_{\sigma\in S+T} \Big( \sum_{\sigma=\alpha+\rho} a_\alpha b_\rho \Big) X^\sigma + \sum_{\sigma\in S+W} \Big( \sum_{\sigma=\alpha+\rho} a_\alpha c_\rho \Big) X^\sigma$$
$$= \ell_1\ell_2 + \ell_1\ell_3. \ \#\#$$

If $M$ has the additional property that $\alpha, \beta \in M$ with $\alpha + \beta = \infty$ implies $\alpha = \infty$ or $\beta = \infty$, then there is a totally ordered group $G$ containing $M \smallsetminus \{\infty\}$ such that $G^+ = M \smallsetminus \{\infty\}$. In this case, $K[[M]]$, denoted $K[[G^+]]$, is a maximal valuation domain with residue field $K$ and value group $G$. [2, Prop. 11.4 and Prop. 11.5].

<u>Theorem 2.4</u>. If $M$ is a value monoid and $K$ is a field, then

there exists a totally ordered group $G$ and an ideal $I$ of $D = K[[G^+]]$ such that $K[[M]] \cong D/I$.

Proof: By Remark 2.1, there is a totally ordered group $G$ and an ideal $I_\infty$ of $G^+ \cup \{\infty\}$ such that $(G^+ \cup \{\infty\})/I_\infty \cong M$. Observe that $I = u^{-1}(I_\infty)$ is an ideal of $D = K[[G^+]]$ where $u: D \longrightarrow G^+ \cup \{\infty\}$ is the semi-value map defined in Chapter 1. If $S \subseteq G^+$ is well-ordered by the induced ordering of $G$, then let $S' = \{\bar{g} \in M;\ g \in S \setminus I_\infty\}$ where the bar denotes the image of an element of $G^+$ in the Rees factor monoid $M$. Define a map $\phi: D \longrightarrow K[[M]]$ by

$$\phi\Big(\sum_{g\in S} a_g X^g\Big) = \begin{cases} \sum\limits_{\bar{g}\in S'} a_{\bar{g}} X^{\bar{g}} & \text{if } S' \neq \phi \\ 0 & \text{if } S' = \phi \end{cases}$$

where $a_{\bar{g}} = a_g$ for $g \notin I_\infty$.

If $\ell_1 = \sum\limits_{\alpha\in S} a_\alpha X^\alpha$ and $\ell_2 = \sum\limits_{\beta\in T} b_\beta X^\beta$ are elements of $D$, then $\phi(\ell_1 + \ell_2) = \sum\limits_{\bar{\gamma}\in(S\cup T)'} c_{\bar{\gamma}} X^{\bar{\gamma}}$ where

$$c_\gamma = \begin{cases} a_\gamma + b_\gamma & \text{if } \gamma \in S \cap T \\ a_\gamma & \text{if } \gamma \in S \setminus T \\ b_\gamma & \text{if } \gamma \in T \setminus S \end{cases}$$

Now $\phi(\ell_1) + \phi(\ell_2) = \sum\limits_{\bar{\alpha}\in S'} a_{\bar{\alpha}} X^{\bar{\alpha}} + \sum\limits_{\bar{\beta}\in T'} b_{\bar{\beta}} X^{\bar{\beta}} = \sum\limits_{\bar{\rho}\in S'\cup T'} c_{\bar{\rho}} X^{\bar{\rho}}$ where

$$c_{\bar{\rho}} = \begin{cases} a_{\bar{\rho}} + b_{\bar{\rho}} & \text{if } \bar{\rho} \in S' \cap T' \\ a_{\bar{\rho}} & \text{if } \bar{\rho} \in S' \setminus T' \\ b_{\bar{\rho}} & \text{if } \bar{\rho} \in T' \setminus S'. \end{cases}$$

Since $S' \cup T' = (S \cup T)'$, $S' \cap T' = (S \cap T)'$, etc.,
$\phi(\ell_1) + \phi(\ell_2) = \phi(\ell_1 + \ell_2)$.

Now $\phi(\ell_1\ell_2) = \phi\left(\sum_{\rho\varepsilon S+T}\left(\sum_{\rho=\alpha+\beta} a_\alpha b_\beta\right)X^\rho\right) = \sum_{\bar\rho\varepsilon(S+T)'}\left(\sum_{\rho=\alpha+\beta} a_\alpha b_\beta\right)X^{\bar\rho}$

while $\phi(\ell_1)\phi(\ell_2) = \left(\sum_{\bar\alpha\varepsilon S'} a_{\bar\alpha}X^{\bar\alpha}\right)\left(\sum_{\bar\beta\varepsilon T'} b_{\bar\beta}X^{\bar\beta}\right) = \sum_{\bar\rho\varepsilon S'+T'}\left(\sum_{\bar\rho=\bar\alpha+\bar\beta} a_{\bar\alpha}b_{\bar\beta}\right)X^{\bar\rho}$.

Observe that $(S + T)' = S' + T'$ because $(S + T)'$
$= \{\bar\rho \in M:\ \rho \in (S + T)\smallsetminus I_\infty\}$
$= \{\bar\rho \in M:\ \rho = \alpha + \beta,\ \alpha \in S,\ \beta \in T,\ \alpha + \beta \notin I_\infty\}$
$= \{\bar\rho \in M:\ \rho = \alpha + \beta,\ \alpha \in S\smallsetminus I_\infty,\ \beta \in T\smallsetminus I_\infty,\ \alpha + \beta \notin I_\infty\}$
$= \{\bar\rho \in M:\ \bar\rho = \bar\alpha + \bar\beta < \infty,\ \alpha \in S\smallsetminus I_\infty,\ \beta \in T\smallsetminus I_\infty\}$
$= \{\bar\rho \in M:\ \bar\rho = \bar\alpha + \bar\beta < \infty,\ \bar\alpha \in S',\ \bar\beta \in T'\} = S' + T'$.

If $\bar\rho$ is a fixed element of $(S + T)'$, then the coefficient of the $X^{\bar\rho}$ term in $\phi(\ell_1\ell_2)$ is the sum of all the products of the form $a_\alpha b_\beta$ where $\alpha \in S$, $\beta \in T$, $\alpha + \beta = \rho$, and $\alpha + \beta \notin I_\infty$. The coefficient of the $X^{\bar\rho}$ term in $\phi(\ell_1)\phi(\ell_2)$ is the sum of all the products of the form $a_{\bar\alpha}b_{\bar\beta}$ where $\bar\rho = \bar\alpha + \bar\beta < \infty$, $\bar\alpha \in S'$, and $\bar\beta \in T'$. But this means $\alpha \in S\smallsetminus I_\infty$, $\beta \in T\smallsetminus I_\infty$ and $\alpha + \beta \notin I_\infty$. So these coefficients are the same. Thus, $\phi(\ell_1\ell_2) = \phi(\ell_1)\phi(\ell_2)$ and $\phi$ is a homomorphism.

Assume $\ell$ is an element of $D$. Then $\ell$ is an element of $I$ if and only if $\text{supp}(\ell) \subseteq I_\infty$. Thus, $\ell \in I$ if and only if $\phi(\ell) = 0$. Hence, the kernel of $\phi$ is $I$ and $D/I \cong K[[M]]$. ##

<u>Corollary 2.5</u>. If $M$ is a value monoid and $K$ is a field, then $K[[M]]$ is a valuation ring with residue field $K$ and value monoid $M$.

<u>Proof</u>: $K[[M]]$ is a valuation ring since the homomorphic image

of a valuation ring is again a valuation ring.

Since the maximal ideal $M'$ of $K[[G^+]]$ consists of all elements $\ell$ such that $0 \notin \mathrm{supp}(\ell)$ [2, Prop. 11.4], the maximal ideal $M$ of $K[[M]]$ is $\Phi(M')$ which consists of all elements $\bar{\ell}$ of $K[[M]]$ such that $0 \notin \mathrm{supp}(\bar{\ell})$. Thus, the residue field of $K[[M]]$ is $K$.

From the definition of $\Phi$, Remark 2.1, and [12, p. 40], we have the following diagram:

$$\begin{array}{rcccl} D & = & K[[G^+]] & \longrightarrow & G^+ \cup \{\infty\} \\ & & \downarrow & & \downarrow \\ D/I & = & K[[M]] & \longrightarrow & (G^+ \cup \{\infty\})/I_\infty \simeq M \end{array}$$

i.e., the value monoid of $K[[M]]$ is $M$. ##

Remark 2.6: We now present a sketch of an argument which shows that 2.3 can be obtained from the proof of 2.4. Consider $K[[M]]$ as a set with the operations of multiplication and addition as defined earlier (see Def. 2.2). $K[[M]]$ has elements 0 and 1, each of which obeys the required identity axiom. The proof of 2.4 establishes a map $\Phi$ from a domain $D$ onto $K[[M]]$ which maps identities to identities and preserves the additive and multiplicative operations. Also, the elements of $D$ which are mapped to 0 of $K[[M]]$ form an ideal. From this, it follows that $K[[M]]$ is a ring, and $\Phi$ is an homomorphism.

In order to show $K[[M]]$ is maximally complete, we require the following lemma.

Lemma 2.7; The homomorphic image of a maximal (2) ring is maximal (2).

Proof: If $R$ is a maximal (2) ring with $I$ being an ideal of $R$, let $\Phi: R \longrightarrow R/I$ be the canonical map. Assume that $\{X \equiv b_\rho \bmod J_\rho\}_{\rho\varepsilon A}$ is a system of congruences of $R/I$ which is finitely solvable in $R/I$ where the $b_\rho$ are elements of $R/I$ and the $J_\rho$ are ideals of $R/I$. Let $I = \Phi^{-1}(J_\rho)$ and pick, for each $\rho$, an element $a_\rho$ of $\Phi^{-1}(b_\rho)$. Form the system of congruences $\{X \equiv a_\rho \bmod I_\rho\}$, and consider a finite system $\{X \equiv a_i \bmod I_i\}$, $i = 1, \ldots, n$. There exists an element $b$ of $R/I$ such that $b \equiv b_i \bmod J_i$, $i = 1, \ldots, n$. If $a$ is an element of $\Phi^{-1}(b)$, then $a - a_i \varepsilon \Phi^{-1}(b - b_i) \subseteq \Phi^{-1}(J_i) = I_i$. Since $\{X \equiv a_\rho \bmod I_\rho\}$ is a finitely solvable system of congruences in a maximal (2) ring, there is an element $a$ of $R$ such that $a \equiv a_\rho \bmod I_\rho$ for all $\rho$. Thus, $\Phi(a) \equiv b_\rho \bmod J_\rho$ for all $\rho$ since $\Phi(a) - b_\rho = \Phi(a - a_\rho)$ which is an element of $\Phi(I_\rho) \subseteq J_\rho$. ##

Theorem 2.8. If $M$ is a value monoid and $K$ is a field, then $K[[M]]$ is a maximal valuation ring and hence, maximally complete.

Proof: By Theorem 2.3 and [2, Prop. 11.5], $K[[M]]$ is a homomorphic image of a maximal valuation ring. By Lemma 2.7, $K[[M]]$ is maximal and by Theorem 1.14, $K[[M]]$ is maximally complete. ##

# CHAPTER III

## EXISTENCE OF A MAXIMAL COMPLETION

In this chapter we show every valuation ring has a maximal completion. Since this argument requires a use of Zorn's Lemma, a bound on the cardinality is needed. First, we require several definitions.

Assume $K$ is a field and $M$ a value monoid. Let $\ell_1 = \sum_{\alpha \varepsilon S} a_\alpha X^\alpha$ and $\ell_2 = \sum_{\beta \varepsilon T} b_\beta X^\beta$ be elements of $K[[M]]$. $\ell_1$ is called an <u>initial segment</u> of $\ell_2$ if $S$ is an initial segment of $T$ (i.e., $S \subseteq T$ and each element of $T \setminus S$ is greater than every element of $S$) and $a_\alpha = b_\alpha$ for all $\alpha \varepsilon S$. $\ell_1$ is a <u>proper initial segment</u> of $\ell_2$ if $\ell_1$ is an initial segment of $\ell_2$ and $\ell_1 \neq \ell_2$. Assume $\{\ell_\sigma\}_{\sigma \varepsilon A}$, $A$ well-ordered, is a <u>sequence of initial segments</u> from $K[[M]]$, which means $\rho < \tau$ for $\rho, \tau \varepsilon A$ implies $\ell_\rho$ is a proper initial segment of $\ell_\tau$. If $S = \bigcup_\sigma S_\sigma$, then $\ell = \sum_{\gamma \varepsilon S} a_\gamma X^\gamma$ is the <u>limit series</u> of $\{\ell_\sigma\}_{\sigma \varepsilon A}$, denoted by $\ell = L(\{\ell_\sigma\})$. Finally, let $C(\ell) = \{\gamma \varepsilon M\colon \ \gamma > \alpha \ \text{ for all } \ \alpha \varepsilon \ \operatorname{supp}(\ell)\}$.

<u>Theorem 3.1</u>. If $R$ is a valuation ring with residue field $k$ and value monoid $M$, then the cardinality of $R$ is bounded by the cardinality of $k[[M]]$.

Proof: The idea is to construct an injective map from $R$ into a subset of $k[[M]]$. For each element $a$ of $k^*$, select an element $s \in U(R)$ such that $\bar{s} = a$, i.e., the residue of $s$ modulo the maximal ideal of $R$ is $a$. For each finite element $\alpha$ of $M$, select an element $x_\alpha$ of $R$ such that $u(x_\alpha) = \alpha$. Let $R_0 = \{sx_\alpha\} \subseteq R$ and $L_0 = \{aX^\alpha\} \subseteq k[[M]]$. Observe that $R_0 \sim L_0$ by $sx_\alpha \leftrightarrow aX^\alpha$ where $\bar{s} = a$.

Now well-order $R \setminus R_0$, say $\{r_1, r_2, \ldots, r_\tau, \ldots\}_{\tau<\Lambda}$. We will find $\{\ell_1, \ell_2, \ldots\} \subseteq k[[M]] \setminus L_0$ so that $\{r_1, r_2, \ldots\} \sim \{\ell_1, \ell_2, \ldots\}$. Set $R_\tau = \{r_1, r_2, \ldots, r_\rho, \ldots\}_{\rho<\tau}$ and $L_\tau = \{\ell_1, \ell_2, \ldots, \ell_\rho, \ldots\}_{\rho<\tau}$. The construction of $\{\ell_1, \ell_2, \ldots\}$ involves transfinite induction where the $\ell_\tau$ will be chosen so that:

(3.2) All proper initial segments of $\ell_\tau$ are elements of $L_0 \cup L_\tau$.

(3.3) If $r$ and $r'$ are distinct elements of $R$ with $\ell$ and $\ell'$ being the corresponding elements in $L_0 \cup \{\ell_1, \ell_2, \ldots\}$, and $\ell - \ell' = aX^\alpha + \ldots$, $a \neq 0$, then $u(r - r' - sx_\alpha) > u(r - r')$ where $sx_\alpha \leftrightarrow aX^\alpha$.

For $L_0$, (3.2) is clearly satisfied, and the following easy argument shows the elements of $R_0$ satisfy (3.3). Assume $r, r' \in R_0$ with $u(r) = u(r')$. Thus, $r = sx_\alpha$ and $r' = s'x_\alpha$ with $sx_\alpha \leftrightarrow aX^\alpha$ and $s'x_\alpha \leftrightarrow a'X^\alpha$. If $s_1$ corresponds to $a - a'$, then $u(sx_\alpha - s'x_\alpha - s_1x_\alpha) > u(sx_\alpha - s'x_\alpha)$ since $s - s' \notin M$ yet $s - s' \equiv s_1 \bmod M$ implies $s - s' - s_1 \in M$. If $u(r) \neq u(r')$, then (3.3) clearly holds.

Defining each $\ell_\tau$ will require a transfinite induction argument in order to construct a sequence of series $\{f_\sigma\}_{\sigma<\lambda} \subseteq k[[M]]$, $\lambda$ a limit ordinal whose cardinality is greater than the cardinality of $M$, such that either (3.4) or (3.5) holds:

(3.4) $f_\sigma \in L_0 \cup L_\tau$ for all $\sigma < \lambda$, and for $\sigma < \rho < \lambda$, $f_\sigma$ is a proper initial segment of $f_\rho$. Also, if $h_\sigma \leftrightarrow f_\sigma$ where $h_\sigma \in R_0 \cup R_\tau$, then $u(r_\tau - h_\sigma) \in C(\{f_\sigma\})$.

(3.5) There exists $\sigma' < \lambda$ such that: $\alpha < \rho \leq \sigma'$ implies $f_\sigma$ is a proper initial segment of $f_\rho$; $f_{\sigma'} \notin L_0 \cup L_\tau$ while $f_\sigma \in L_0 \cup L_\tau$ for $\sigma < \alpha'$; $\sigma' < \sigma$ implies $f_{\sigma'} = f_\sigma$; and finally, for $\sigma < \sigma'$, $u(r_\tau - h_\sigma) \in C(f_\sigma)$.

In the case of (3.5), $\ell_\tau$ will be $f_{\sigma'}$; in the case of (3.4), $\ell_\tau$ will be $L(\{f_\sigma\})$.

Now consider $r_1$. We can write $r_1 = s_1x_{\alpha_1} + m_1$ where $m_1 \in M$, $u(m_1) > \alpha_1$ and $s_1x_{\alpha_1} \in R_0$. Set $f_1 = a_1X^{\alpha_1}$ where $s_1x_{\alpha_1} \leftrightarrow a_1X^{\alpha_1}$. Now write $r_1 - s_1x_{\alpha_1} = s_2x_{\alpha_2} + m_2$ where $m_2 \in M$, $u(m_2) > \alpha_2$ and $s_2x_{\alpha_2} \in R_0$. Let $f_2 = f_1 + a_2X^{\alpha_2}$ where $s_2x_{\alpha_2} \leftrightarrow a_2X^{\alpha_2}$. Since $f_2 \notin L_0$ and $u(r_1 - h_1) \in C(f_1)$, we have (3.5) holding after setting $f_\sigma = f_2$ for $2 < \sigma < \lambda$. Since all proper initial segments of $\ell_1$ are clearly in $L_0$, (3.2) is satisfied. Finally, we need to show if $r$ and $r'$ are distinct elements of $R_0 \cup R_1$, then (3.3) is satisfied. Actually, we need only to consider the case when $r = sx_{\alpha_1}$ and $r' = r_1$. Now $aX^{\alpha_1} - \ell_1 = (a - a_1)X^{\alpha_1} + a_2X^{\alpha_2}$. Let

$s'x_{\alpha_1} \leftrightarrow (a - a_1)X^{\alpha_1}$. So $u(r - r' - s'x_{\alpha_1}) > u(r - r')$ since $s - s_1 \notin M$, and $s - s_1 - s' \in M$ because $s - s_1 \equiv s' \bmod M$.

Assume $\ell_1, \ell_2, \ldots, \ell_\rho, \ldots$ have been defined for $\rho < \tau$ where $1 < \tau < \Lambda$. Now consider $r_\tau$; we need to construct $\ell_\tau$. Write $r_3 = s_3x_{\alpha_3} + m_3$ where $m_3 \in M$, $u(m_3) > \alpha_3$ and $s_3x_{\alpha_3} \in R_0$. Set $f_1' = a_3X^{\alpha_3}$ where $s_3x_{\alpha_3} \leftrightarrow a_3X^{\alpha_3}$. Note that $u(r_\tau - h_1') \in C(f_1')$. Assume $\{f_\sigma'\}_{\sigma<\zeta}$ has been defined where: $\zeta < \lambda$; $f_\sigma' \in L_0 \cup L_\tau$; $\sigma < \delta < \zeta$ implies $f_\sigma'$ is a proper initial segment of $f_\delta'$; and, if $h_\sigma' \leftrightarrow f_\sigma'$, then $u(r_\tau - h_\sigma') \in C(f_\sigma')$.

If $\zeta$ is not a limit ordinal, then write $r_\tau - h_{\zeta-1}' = s_4x_{\alpha_4} + m_4$ where $s_4x_{\alpha_4} \in R_0$, $m_4 \in M$ and $u(m_4) > \alpha_4$. Set $f_\zeta' = f_{\zeta-1}' + a_4X^{\alpha_4}$ where $s_4x_{\alpha_4} \leftrightarrow a_4X^{\alpha_4} \in L_0$. If $f_\zeta' \in L_0 \cup L_\tau$, then there exists an element $h_\zeta'$ of $R_0 \cup R_\tau$ such that $h_\zeta' \leftrightarrow f_\zeta'$. We now show $u(r_\tau - h_\zeta') \in C(f_\zeta')$. Since $f_\zeta' - f_{\zeta-1}' = a_4X^{\alpha_4}$, $u(h_\zeta' - h_{\zeta-1}' - s_4x_{\alpha_4}) > u(h_\zeta' - h_{\zeta-1}')$ which implies $h_\zeta' - h_{\zeta-1}' = sx_{\alpha_4} + m$ where $s \equiv s_4 \bmod M$, $m \in M$ and $u(m) > \alpha_4$. Thus, $u(r_\tau - h_\zeta') = u\big((s_4 - s)x_{\alpha_4} + m\big) = u(m) > \alpha_4$, i.e., $u(r_\tau - h_\zeta') \in C(f_\zeta')$. However, if $f_\zeta' \notin L_0 \cup L_\tau$, then set $f_\sigma' = f_\zeta'$ for $\zeta < \sigma < \lambda$. In particular, (3.5) occurs and $\ell_\tau = f_\zeta'$.

If $\zeta$ is a limit ordinal, then set $f_\zeta' = L(\{f_\sigma'\}_{\sigma<\zeta})$. If $f_\zeta' \in L_0 \cup L_\tau$, then there exists $h_\zeta' \in R_0 \cup R_\tau$ such that $h_\zeta' \leftrightarrow f_\zeta'$. We claim that $u(r_\tau - h_\zeta')$ is an element of $C(f_\zeta')$.

Since $f'_\sigma$ is an initial segment of $f'_\zeta$ for $\sigma < \zeta$, $u(f'_\zeta - f'_\sigma) \in C(f'_\sigma)$. If $f'_\zeta - f'_\sigma = aX^\alpha + \dots$, then $u(h'_\zeta - h'_\sigma - sx_\alpha) > u(h'_\zeta - h'_\sigma)$ where $sx_\alpha \leftrightarrow aX^\alpha$. Hence $\alpha = u(h'_\zeta - h'_\sigma) \in C(f'_\sigma)$. Since $u(r_\tau - h'_\sigma)$ and $u(h'_\zeta - h'_\sigma)$ are elements of $C(f'_\sigma)$, $u(r_\tau - h'_\zeta) = u(r_\tau - h'_\sigma + h'_\sigma - h'_\zeta) \in C(f'_\sigma)$. But this holds for all $\sigma < \zeta$, hence, $u(r_\tau - h'_\zeta) \in C(f'_\zeta)$. If $f'_\zeta \notin L_0 \cup L_\tau$, then set $f'_\sigma = f'_\zeta$ for $\zeta < \sigma < \lambda$, i.e., (3.5) occurs.

Thus, we have $\{f'_\sigma\}_{\sigma<\lambda}$ constructed according to the desired specifications. If (3.5) occurs, then $\ell_\tau = f'_{\sigma'}$, and $\ell_\tau \notin L_0 \cup L_\tau$ by definition of (3.5). If (3.4) occurs, then set $\ell_\tau = L(\{f'_\sigma\})$. We now show $\ell_\tau \notin L_0 \cup L_\tau$. Note that $\bigcap_\sigma C(\{f'_\sigma\}) = \phi$, for if not, then we could find a subset of $M$ of cardinality greater than $M$ (because of the choice of $\lambda$). Since $\bigcap C(\{f'_\sigma\}) = \phi$ and $u(r_\tau - h'_\sigma) \in C(f'_\sigma)$ for $\sigma < \lambda$, if there is an element $h \in R_0 \cup R_\tau$ such that $h \leftrightarrow \ell_\tau$, then $u(r_\tau - h) = \infty$, i.e., $h = r_\tau$. This contradicts $r_\tau \notin R_0 \cup R_\tau$; thus, there is no such $h$. Hence, $\ell_\tau \notin L_0 \cup L_\tau$.

We now show if $f$ is a proper initial segment of $\ell_\tau$, then $f \in L_0 \cup L_\tau$, i.e. $\ell_\tau$ satisfies (3.2). Let $C_\sigma = C(f'_\sigma) \smallsetminus C(\ell_\tau)$ and $C = C(f) \smallsetminus C(\ell_\tau)$.

First, if (3.4) holds, then observe that $\bigcap C_\sigma = \phi$. If there exists $\psi < \lambda$ such that $C_\psi \subseteq C$, then $f$ is an initial segment of $f_\psi$ which implies $f \in L_0 \cup L_\tau$. If there exists no such $\psi$, then $C \subseteq \bigcap C_\sigma = \phi$ which contradicts the choice of $f$ as a proper initial segment of $\ell_\tau$.

Next, if (3.5) holds and $\sigma'$ is a limit ordinal, then repeat the argument of the above paragraph using $\sigma < \sigma'$.

Finally, if (3.5) holds and $\sigma'$ is not a limit ordinal, then $\bigcap_{\sigma<\sigma'} C_\sigma = C_{\sigma'-1}$, which consists of only one element. If there exists $\psi < \sigma'$ such that $C_\psi \subseteq C$, then $f$ is an initial segment of $f_\psi$ which implies $f \in L_0 \cup L_\tau$. If there exists no such $\psi$, then $C \subseteq \bigcap_{\sigma<\sigma'} C_\sigma$. Thus, $C = C_{\sigma'-1}$ and $f = f'_{\sigma'-1}$. Therefore, $L_0 \cup L_{\tau+1}$ satisfies (3.2).

By extending the arguments in the preceding three paragraphs, it is possible to show every initial segment $f$ of $\ell_\tau$ appears among the $f'_\sigma$ used in defining $\ell_\tau$. If (3.4) holds, then let $\psi$ be the smallest $\sigma$ such that $C_\sigma \subseteq C$. If $\psi = 1$, then $C_1 = C$ which implies $f = f'_1$. If $\psi > 1$ and $\psi$ is not a limit ordinal, then $C_\psi \subseteq C \subset C_{\psi-1}$. But by the construction of $f'_\psi$ from $f'_{\psi-1}$, $C_{\psi-1} \setminus C_\psi$ contains only one element. Thus, $C_\psi = C$ and $f = f_\psi$. If $\psi$ is a limit ordinal, then $C_\psi \subseteq C \subset C_\sigma$ for all $\sigma < \psi$. Hence, $C \subseteq \bigcap_{\sigma<\psi} C_\sigma = C_\psi$, which implies $f = f_\psi$. In the case of (3.5), the argument is similar.

To show (3.3) holds for $R_0 \cup R_{\tau+1}$, we first show (3.3) holds for $r_\tau$ and the element $h \in R_0 \cup R_\tau$ where $h \leftrightarrow f$, $f$ a proper initial segment of $\ell_\tau$. By the previous paragraph, $f$ appears among the $f'_\sigma$ used in defining $\ell_\tau$, say $f = f'_\psi$ (so $h = h'_\psi$). Then $f'_{\psi+1} = f'_\psi + aX^\alpha$ with $sx_\alpha \leftrightarrow aX^\alpha$ where $r_\tau - h = sx_\alpha + m$, $u(m) > \alpha$. Since $\ell_\tau - f = aX^\alpha + \dots$, $u(r_\tau - h - sx_\alpha) > u(r_\tau - h)$, i.e., (3.3) holds.

To finish the proof of (3.3) holding for $R_0 \cup R_{\tau+1}$, it suffices to consider $r_\tau$ and an arbitrary element $r$ of $R_0 \cup R_\tau$ where $u(r) = u(r_\tau)$. Let $\ell$ be an element of $L_0 \cup L_\tau$ such that $r \leftrightarrow \ell$. Assume that $\ell = f + bX^\beta + \ldots$ and $\ell_\tau = f + aX^\alpha + \ldots$ where $bX^\beta \neq aX^\alpha$. If $f = 0$, then $\alpha = \beta$, and $r = tx_\alpha + m_5$ and $r_\tau = sx_\alpha + m_6$ with $u(m_5), u(m_6) > \alpha$. Let $s'x_\alpha \leftrightarrow (b - a)X^\alpha$. Since $t - s \notin M$ while $t - s \equiv s' \bmod M$, $u(r - r_\tau - s'x_\alpha) > u(r - r_\tau)$, i.e., (3.3) holds. We now treat the case of $f \neq 0$. Then let $h$ be the element of $R_0 \cup R_\tau$ such that $h \leftrightarrow f$. Since $r$ and $h$ are elements of $R_0 \cup R_\tau$, and $R_0 \cup R_\tau$ satisfies (3.3), then $u(r - h - tx_\beta) > u(r - h)$ where $tx_\beta \leftrightarrow bX^\beta$. By the previous paragraph, $u(r_\tau - h - sx_\alpha) > u(r_\tau - h)$ where $sx_\alpha \leftrightarrow aX^\alpha$. Therefore, $r - h = t'x_\beta + m'$ where $t' \equiv t \bmod M$ and $u(m') > \beta$; and, $r_\tau - h = s'x_\alpha + m''$ where $s' \equiv s \bmod M$ and $u(m'') > \alpha$. There are three cases to consider: 1.) If $\alpha = \beta$, then let $s''x_\alpha$ be the element of $R_0 \cup R_\tau$ such that $s''x_\alpha \leftrightarrow (b - a)X^\alpha$. Thus, $u(r - r_\tau - s''x_\alpha) > u(r - r_\tau)$ since $\overline{s''} = b - a = \overline{t - s} = \overline{t' - s'}$, $r - r_\tau = (t' - s')x_\alpha + \ldots$, and $t' - s' \notin M$. 2.) If $\beta < \alpha$, then $u(r - r_\tau - tx_\beta) > u(r - r_\tau)$ since $r - r_\tau = t'x_\beta + \ldots$ and $t' \equiv t \bmod M$. 3.) If $\alpha < \beta$, then the argument is identical to 2.). Therefore, (3.3) holds for $R_0 \cup R_{\tau+1}$.

Hence, a transfinite construction yields $\{\ell_1, \ell_2, \ldots\} \subseteq k[[M]]$ with $\{r_1, r_2, \ldots\} \sim \{\ell_1, \ell_2, \ldots\}$. ##

Note that the above proof is a slight variation (and elaboration) of the proof for the case of valuation domains. See [11, Lemma 5,

p. 37].

It is now possible to show that every valuation ring has a maximal immediate extension, i.e., a maximal completion.

Assume that $R$ is a valuation ring with value monoid $M$ and residue field $k$. Let $S$ be a set such that $R \subset S$ with the cardinality of $S$ being at least the cardinality of $\mathcal{P}(k[[M]])$, the power set of $k[[M]]$. $R$ can be identified with an element of $\mathcal{S} = \mathcal{P}(S) \times \mathcal{P}(S \times S \times S) \times \mathcal{P}(S \times S \times S)$ in the following manner. By using the ring structure of $R$, it is possible to identify the operations of addition and multiplication on $R$ with subsets $A$ and $P$ of $R \times R \times R \subset S \times S \times S$, i.e., $A = \{(r, r', r + r'): r, r' \in R\}$ and $P = \{(r, r', rr'): r, r' \in R\}$. Thus, $R$ is identified with $\hat{R} = (R, A, P) \in \mathcal{S}$.

Given an element $R_\rho = (S_\rho, A_\rho, P_\rho)$ of $\mathcal{S}$ with $A_\rho, P_\rho \subset S_\rho \times S_\rho \times S_\rho$, $A_\rho$ can be used to define an addition operation on $S_\rho$ by $s_1 + s_2 = s_3$ provided $(s_1, s_2, s_3) \in A_\rho$. Similarly, $P_\rho$ can be used to define a product. $R_\rho$ is a ring provided these operations are defined for every pair of elements from $S_\rho$ and the operations satisfy the required ring axioms.

If $R_1 = (S_1, A_1, P_1)$ and $R_2 = (S_2, A_2, P_2)$ are elements of $\mathcal{S}$, then $R_1$ is said to be contained in $R_2$, written $R_1 \subseteq R_2$, provided $S_1 \subseteq S_2$, $A_1 \subseteq A_2$, and $P_1 \subseteq P_2$.

Let $\mathcal{V}$ be the set of valuation rings in $\mathcal{S}$ which are immediate extensions of $\hat{R} = (R, A, P)$. (Note that if $R_\rho = (S_\rho, A_\rho, P_\rho) \in \mathcal{V}$, then $S_\rho$ is an immediate extension of $R$.) $\mathcal{V}$ is partially ordered by containment. Assume that $\mathcal{V}_1 = \{(S_\rho, A_\rho, P_\rho)\}$ is a totally ordered subset of $V$. Since it is clear that $R_1 = \bigcup_\rho V_1 = (S_1, A_1, P_1)$ is a valuation ring, $R_1$ is an upper bound for $V_1$ provided $R_1$ is an immediate extension of $\hat{R}$.

To see that $R_1$ is an immediate extension of $\hat{R}$, we first show that $M_1 \cap R = \hat{M}$. If $r \in \hat{M}$, then $r \in M_\rho$ for all $R_\rho \in V_1$. This implies $r \in M_1 \cap R$. If $r \in M_1 \cap R$, then $r \in S_\rho$ for some $R_\rho \in V_1$. Since $r \in U(R_\rho)$ implies $r \in U(R_1)$, $r \in M_\rho$. Hence, $r \in M_\rho \cap R = \hat{M}$. Next, we show $k_1 = \hat{k}$. Since $M_1 \cap R = \hat{M}$, we need only to show the canonical mapping $\hat{k} \longrightarrow k_1$ is a surjection. If $s_1 \in U(R_1)$, then there exists an element $R_\rho \in V_1$ such that $s_1 \in U(R_\rho)$. Since $k_\rho = \hat{k}$, there exists an element $s$ of $U(R)$ and $m \in M_\rho \subseteq M_1$ such that $s_1 = s + m$. Finally, we show $M_1 = \hat{M}$. We need only show the canonical map $\hat{M} \longrightarrow M_1$ is a surjection since $M_1 \cap R = \hat{M}$ with $R$ being a valuation ring implies the canonical map is an injection. If $I$ is a principal ideal of $R_1$ generated by $r_1$, then since $r_1 \in S_\rho$ for some $R_\rho \in V_1$ and $M_\rho = M$, there exists an element $r \in R$ and $s_\rho \in U(R_\rho)$ such that $r = r_\rho s_1$, i.e., $rS_1 = I$.

By Zorn's Lemma, $V$ has a maximal element $R_\lambda = (S_\lambda, A_\lambda, P_\lambda)$. If $R'$ is an immediate extension of $R_\lambda$, then $R'$ can be identified with an element of $V$ as follows. Because of the choice of $S$ and Lemma 3.1, there is an injection $\Phi: R' \longrightarrow S$ where $\Phi$ restricted to $S_\lambda$ is the identity. Thus, $R'$ is identified with $(\Phi(R'), A', P')$ where $A'$, $P' \subseteq \Phi(R') \times \Phi(R') \times \Phi(R')$ are defined in the obvious manner to reflect the additive and multiplicative operations on $R'$. This implies $(\Phi(R'), A', P') \in S$ and is an immediate extension of $\hat{R}$, i.e., $(\Phi(R'),A',P') \in \mathcal{V}$. Since $R_\lambda$ is a maximal element of $\mathcal{V}$, $R_\lambda = (\Phi(R'),A',P')$, i.e., $R_\lambda$ has no proper immediate extensions. Thus, $R_\lambda = (S_\lambda,A_\lambda,P_\lambda)$ is a maximal completion of $\hat{R} = (R,A,P)$ which implies $S_\lambda$ is a maximal completion of $R$. Therefore, we have the following theorem:

<u>Theorem 3.6</u>. If $R$ is a valuation ring, then $R$ has a maximal completion.

# CHAPTER IV

## REMARKS

The goal of this work was a uniqueness theorem for maximal completions of certain types of valuation rings. The idea was that, under suitable conditions, a maximal completion of a valuation ring would be a long power series ring, which is a homomorphic image of a valuation domain. Thus, we would have a class of valuation rings, each of which would be contained in a homomorphic image of a valuation domain. Since the converse of 1.15 remains unproven, attention was shifted to maximal valuation rings. The hope was that, under suitable assumptions, a maximal valuation ring would be a long power series ring and thus, a homomorphic image of a valuation domain. However, a very important tool for working with valuation domains does not seem to apply to valuation rings (see Question 4.7). Before expanding on these ideas, we review the procedure followed by Kaplansky in his work [5] and [6] on maximally complete valuation domains (valued fields).

Let $D$ be a valuation domain with field of fractions $K$, residue field $k$ and value group $G$. $D$ is said to satisfy <u>hypothesis A</u> if the characteristic of $k$ is $0$, or if the characteristic of $k$ is $p > 0$, $pG = G$, and any equation of the form

$X^{p^n} + a_1 X^{p^{n-1}} + \ldots + a_{n-1} X^p + a_n X + a_{n+1} = 0$, with coefficients from $k$, has a root in $k$. Kaplansky was able to prove: If D is a valuation domain which satisfies hypothesis A, then the maximal completion of D is uniquely determined up to analytical equivalence over D. [5, Thm. 5]. Analytical equivalence (over D) means an isomorphism which preserves values (and is the identity on D). If D is maximally complete and satisfies hypothesis A with the characteristic of $k$ equaling the characteristic of K, then D is analytically isomorphic to the twisted long power series domain $k[[G^+, c_{\alpha,\beta}]]$. [5, Thm. 6]. Note that $c_{\alpha,\beta}$ is a factor set and multiplication in the twisted long power series domain is defined by $X^\alpha X^\beta = c_{\alpha,\beta} X^{\alpha+\beta}$ with $c_{\alpha,\beta}$ being an element of $k$.

Kaplansky was able to obtain the above two results by using pseudo-convergent sequences. If D' and D" are immediate extensions of D with field of fractions K' and K" respectively, then K' and K" can be obtained from K by adjoining limits of pseudo-convergent sequences. Knowing when these steps are unique is the key to obtaining results about the uniqueness and structure of maximal completions. A pseudo-convergent sequence $\{a_\rho\}_{\rho\epsilon A}$ in K is said to be of algebraic type (over K) if there exists some polynomial $f$ belonging to K[X] such that $u(f(a_\rho)) < u(f(a_\sigma))$ for sufficiently large $\rho < \sigma$. A pseudo-convergent sequence is said to be of transcendental type (over K) if for all $\rho$ and $\sigma$ sufficiently large, $u(f(a_\rho)) = u(f(a_\sigma))$ for all $f \in K[X]$. Because of Proposition 4.1 and Lemma 1.7, it follows that if $\{a_\rho\}_{\rho\epsilon A}$ is a pseudo-convergent sequence in K, then $\{a_\rho\}_{\rho\epsilon A}$ is either of algebraic or transcendental type.

If there is a transcendental pseudo-convergent sequence

$\{a_\rho\}_{\rho \varepsilon A}$ in $K$ without a pseudo-limit in $K$, then there exists an immediate transcendental extension $K(z)$ of $K$ with $z$ being a pseudo-limit of $\{a_\rho\}_{\rho \varepsilon A}$. The valuation on $K(z)$ is defined by: the value of $f(z)$ is the value which $u(f(a_\rho))$ eventually assumes. If $\{a_\rho\}_{\rho \varepsilon A}$ is of algebraic type in $K$ without a limit in $K$, then there is an immediate algebraic extension $K(z)$ of $K$ with $z$ being a pseudo-limit of $\{a_\rho\}_{\rho \varepsilon A}$. Among the polynomials for which $u(f(a_\rho)) < u(f(a_\sigma))$ choose one of least degree, say $h(X)$. Let $z$ be a root of $h(X) = 0$ and define the value of any polynomial $f$ of degree less than the degree of $h$ to be the value which $u(f(a_\rho))$ eventually assumes. [5, Thms. 2 and 3]. The transcendental immediate extensions are uniquely determined by this process. However, if $K(y)$ is an immediate extension of $K$ where $y$ is a pseudo-limit of an algebraic pseudo-convergent sequence, then $K(y)$ is not necessarily analytically isomorphic over $K$ to the $K(z)$ defined above. They are analytically equivalent provided $y$ is a root of $h(X) = 0$.

Seeing that a pseudo-convergent sequence is either of transcendental or algebraic type relies on the following proposition:

<u>Proposition 4.1</u>. If $\{a_\rho\}_{\rho \varepsilon A}$ is a pseudo-convergent sequence in the valued field $K$, then for any non-constant polynomial $f$ belonging to $K[X]$, $\{f(a_\rho)\}_{\rho \varepsilon A}$ is eventually pseudo-convergent.

<u>Proof</u>: [11, Lemma 9, p. 40]. ##

The proof of 4.1 relies on the following lemma; we present a simpler proof than the one found in [11, Lemma 8, p. 40].

<u>Lemma 4.2</u>. Assume that $\beta_1, \ldots, \beta_n$ are elements of a

totally ordered abelian group $G$. Let $\{\alpha_\rho\}_{\rho\varepsilon A}$, $A$ well-ordered, be a monotonically increasing set of elements of $G$ without a last element. Let $t_1, \ldots, t_n$ be distinct positive integers. Then there is $\sigma \varepsilon A$ and $1 \leq n' \leq n$ such that $\beta_{n'} + t_{n'}\alpha_\rho < \beta_i + t_i\alpha_\rho$ for $i \neq n'$ and $\rho > \sigma$.

<u>Proof</u>: For fixed $i \neq j$, assume $t_j - t_i > 0$ and compare $\beta_i - \beta_j$ with $(t_j - t_i)\alpha_\rho$. Since $\{\alpha_\rho\}_{\rho\varepsilon A}$ is monotonically increasing, either $\beta_i - \beta_j > (t_j - t_i)\alpha_\rho$ for all $\rho$ or there exists $\sigma' \varepsilon A$ such that $\beta_i - \beta_j < (t_j - t_i)\alpha_\rho$ for $\rho > \sigma'$. If $t_i - t_j > 0$, then compare $\beta_j - \beta_i$ with $(t_i - t_j)\alpha_\rho$. In any event, there exists $\sigma' \varepsilon A$ such that $\rho > \sigma'$ implies $\beta_i + t_i\alpha_\rho < \beta_j + t_j\alpha_\rho$ or vice versa. The conclusion of the lemma then follows. ##

Using Remark 2.1, we can obtain a result similar to 4.2 for value monoids.

<u>Lemma 4.3</u>. Let $\bar{\beta}_1, \ldots, \bar{\beta}_n$ be finite elements of a value monoid $M$ and $\{\bar{\alpha}_\rho\}_{\rho\varepsilon A}$, $A$ well-ordered, a monotonically increasing set of elements of $M$ without a last element. If $t_1, \ldots, t_n$ are distinct positive integers, then either there exists $\sigma \varepsilon A$ such that $\rho > \sigma$ implies $\beta_i + t_i\alpha_\rho = \infty$ for all $i$ or there exists $\sigma$ and $1 \leq n' \leq n$ such that $\beta_{n'} + t_{n'}\alpha_\rho < \beta_i + t_i\alpha_\rho$ for $\rho > \sigma$ and $i \neq n'$.

<u>Proof</u>: By Remark 2.1, there is a totally ordered group $G$ and an upper segment $I_\infty$ of $G^+ \cup \{\infty\}$ such that $(G^+ \cup \{\infty\})/I_\infty$ is isomorphic to $M$. Let $\beta_1, \ldots, \beta_n$ be the elements of $G^+$ which

map to $\bar{\beta}_1, \ldots, \bar{\beta}_n$ respectively; let $\{\alpha_\rho\}_{\rho\varepsilon A}$ be the sequence in $G^+$ which maps to $\{\bar{\alpha}_\rho\}_{\rho\varepsilon A}$. By Lemma 4.2, there is an element $\sigma \varepsilon A$ and $1 \leq n' \leq n$ such that $\beta_i + t_i\alpha_\rho > \beta_{n'} + t_{n'}\alpha_\rho$ for $\rho > \sigma$ and $i \neq n'$. If $\beta_{n'} + t_{n'}\alpha_\rho$ is not a member of $I_\infty$ for all $\rho$, then $\bar{\beta}_i + t_i\bar{\alpha}_\rho > \bar{\beta}_{n'} + t_{n'}\bar{\alpha}_\rho$ for $\rho > \sigma$ and $i \neq n'$. If $\beta_{n'} + t_{n'}\alpha_{\rho'}$ is an element of $I$ for some $\rho'$, then $\beta_i + t_i\alpha_\rho$ is an element of $I_\infty$ for $\rho > \rho'$ and $i = 1, \ldots, n$. Thus, $\infty = \bar{\beta}_i + t_i\bar{\alpha}_\rho$ for $\rho > \rho'$ and all $i$. ##

After proving Lemma 4.4, we can obtain a result for valuation rings for which 4.1 is a special case. The proof of Proposition 4.5 presented below is simpler than the one in [11, Lemma 9, p. 40].

Lemma 4.4. Assume that $\{a_\rho\}_{\rho\varepsilon A}$, $A$ well-ordered, is a sequence in a valuation ring $R$ such that $\{u(a_\rho - a)\}_{\rho\varepsilon A}$ is monotonically increasing for some element $a$ of $R$. Then $\{a_\rho\}_{\rho\varepsilon A}$ is a pseudo-convergent sequence which has $a$ as a pseudo-limit.

Proof: This follows immediately from
$u(a_\tau - a_\rho) = u(a_\tau - a + a - a_\rho) = u(a - a_\rho)$ for $\rho > \tau$. ##

Proposition 4.5. Let $\{\alpha_\rho\}_{\rho\varepsilon A}$ be a pseudo-convergent sequence in a valuation ring $R$. If $R$ has an immediate extension $R'$ such that $\{a_\rho\}_{\rho\varepsilon A}$ has a pseudo-limit $a$ in $R'$, then for any non-constant polynomial $f(X)$ from $R[X]$ either eventually $\{f(a_\rho)\}_{\rho\varepsilon A}$ is pseudo-convergent or eventually $f(a_\rho) = f(a)$.

Proof: Note that if $K$ is a valued field, then every pseudo-convergent sequence from $K$ will have a pseudo-limit in a maximal completion of $K$. Since $K[X]$ is a unique factorization domain, it is impossible for the latter conclusion to hold.

Consider the Taylor expansion of $f(X)$ about $a$: $f(X) = f(a) + f'(a)(X - a) + (f''(a)/(2!))(X - a)^2 + \dots$ . If $\beta_i = f^{(i)}(a)/i!$, then $u'(f(a_\rho) - f(a)) \geq \inf \{\beta_i + i\gamma_\rho\}$. By Lemma 4.3, if there exists an $i$ such that $\beta_i + i\gamma_\rho < \infty$ for all $\rho$, then there exists $\sigma$ and $j$ such that $\beta_j + j\gamma_\rho < \beta_i + i\gamma_\rho$ for $i \neq j$ and $\rho > \sigma$. This implies $u'(f(a_\rho) - f(a)) = \beta_j + j\gamma_\rho$ for $\rho > \sigma$. Since $\{\beta_j + j\gamma_\rho\}_{\rho>\sigma}$ is monotonically increasing, $\{f(a_\rho)\}_{\rho>\sigma}$ is pseudo-convergent in $R'$ (also pseudo-convergent in $R$) with pseudo-limit $f(a)$ by Lemma 4.4. If for each $i$, $\beta_i + i\gamma_\rho$ is eventually $\infty$, then there is some $\sigma$ such that $\rho > \sigma$ implies $u'(f(a_\rho) - f(a)) = \infty$, i.e., $f(a_\rho) = f(a)$ for $\rho > \sigma$. ##

We now present several unanswered questions which developed from working on structure theorems for maximally complete (or maximal) valuation rings.

Question 4.6. Is a valuation ring which is maximally complete also maximal? (This is the converse of 1.12.)

Showing that a maximally complete valuation domain is a maximal valuation domain relies on the construction of immediate extensions using pseudo-limits of algebraic and transcendental pseudo-convergent sequences [2, pp. 93-95]. The above facts (Lemma 4.4 and Proposition 4.5) about pseudo-convergent sequences in valuation rings seem to have no applications to the valuation ring case. The key appears to be an answer to the following:

Question 4.7. Given a valuation ring (domain) $R$, how can one construct an immediate extension $R'$ of $R$?

In the domain case, Question 4.7 is answered using the quotient field of a valuation domain. But can one avoid the use of the quotient field?

Question 4.8. i.) Under what conditions is a maximal completion of a valuation ring unique? ii.) When are maximal rings which are also immediate extensions of a valuation ring $R$ uniquely determined?

Of course, in order to show that the maximal completion of a valuation domain $D$ is $K[[G^+]]$, one must first show that $D$ contains the monoid ring $K[X^{G^+}]$.

Question 4.9. i.) When is the ring $k[X^M]/(X^\infty)$ contained in a valuation ring $R$? ii.) If the valuation ring $R$ contains $k[X^M]/(X^\infty)$, is every maximal completion of $R$ analytically equivalent (over $R$) to $k[[M]]$?

If the domain case is used as a guide, the answer to 4.7 would provide answers to 4.6, 4.8 and 4.9 ii.). As mentioned before, since 4.6 is unknown, our approach shifted to using maximal valuation rings. However, even trying to show the following failed. Suppose $R$ is a 0-dimensional valuation ring with residue field $k$, of characteristic 0, and value monoid $M$. If $R$ is maximal and contains $k[X^M]/(X^\infty)$, then $R$ is isomorphic to $k[[M]]$.

Question 4.10. If the valuation ring $R$ is 0-dimensional, is

R a homomorphic image of a rank 1 valuation domain?

Question 4.11. Assume R is a quasi-local ring. i.) If R is maximal (1), then is R maximal (2)? (See Proposition 1.12.) ii.) Does R being maximal (1) imply that R is maximally complete? (See Theorem 1.15.) iii.) If R is maximally complete, is R maximal (2)? (See Question 4.6.)

Question 4.12. Can Theorems 3.1 and 3.6 be extended to quasi-local rings? In other words, is there a bound on the cardinality of an arbitrary quasi-local ring R which depends only on the cardinalities of the semi-value monoid (R) and residue field $k$? Does every quasi-local ring have a maximal completion?

We conclude this chapter with some observations about 0-dimensional valuation rings. Recall that the value monoid of a valuation ring is isomorphic to a Rees factor monoid $(G^+ \cup \{\infty\})/I_\infty$ where G is a totally ordered abelian group and $I_\infty$ is an upper segment of $G^+ \cup \{\infty\}$. If the valuation ring is 0-dimensional, then the following shows that G is isomorphic to a subgroup of the real numbers.

If R is a 0-dimensional valuation ring, then every nonzero nonunit of R is nilpotent. As a result, every nonzero element of the value monoid M has finite index, i.e., if $\alpha$ is an element of M, then there exists a positive integer n such that $n\alpha = \infty$. Thus, $M \setminus \{0\}$ satisfies the requirements of what Clifford calls a segment [3, p. 636]. The 0-segmental monoids considered by Shores [12, p. 406] were not segments, yet everything in Section 3 of [3] held except for the results dealing with the archimedean property. But since $M \setminus \{0\}$ is a segment, those results concerning the archimedean property do apply to $M \setminus \{0\}$. (A naturally totally ordered monoid $S$ is <u>archimedean</u> if for each pair of elements $a$ and $b$ of $S$ with $b > 0$, there exists a positive integer $n$ such that $nb \geq a$.) Thus, we have:

Lemma 4.13. If $M$ is a value monoid such that every nonzero element of $M$ has finite index, then there is a naturally totally ordered, archimedean, cancellative monoid $S$ and an upper segment $S_\infty$ of $S$ such that $M$ is isomorphic to the Rees factor monoid $S/S_\infty$.

But a naturally totally ordered, archimedean, cancellative monoid can be embedded (as precisely the nonnegative elements) in a totally ordered archimedean group $G$. Since a totally ordered archimedean group is isomorphic to a subgroup of the real numbers, we have:

Proposition 4.14. If $M$ is a value monoid such that every nonzero element of $M$ has finite index, then there is a subgroup $G$ of the real numbers and there is an upper segment $I_\infty$ of $G^+ \cup \{\infty\}$ such that $M$ is isomorphic to $(G^+ \cup \{\infty\})/I_\infty$.

Observe that if Question 4.10 has an affirmative answer, then Proposition 4.14 follows immediately.

The next proposition shows that an immediate extension of a 0-dimensional valuation ring $R$ contains no elements which are transcendental over $R$. Recall that the transcendental immediate extensions for the domain case are the easiest to handle.

<u>Proposition 4.15</u>. If $R'$ is an immediate extension of the 0-dimensional valuation ring $R$, then $R'$ is integral over $R$.

<u>Proof</u>: Since $R'$ is also 0-dimensional, every nonzero nonunit of $R'$ is nilpotent. Assume $a$ is an element of $U(R')$ $R$. Since $k = k'$, there is an element $a_1$ of $U(R)$ such that $\bar{a} = \bar{a}_1$, i.e., $a = a_1 + m$ where $m \neq 0$ belongs to $\mathcal{M}'$. If $f(X) = (X - a_1)^r$ where $r$ is a positive integer such that $m^r = 0$, then $f(a) = (a - a_1)^r = m^r = 0$. ##

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